Classical Integrability of Non Abelian Affine Toda Models[1]

J.F. Gomes[a], E. P. Gueuvoghlanian[a], G.M. Sotkov[a] and A.H. Zimerman[a]

[a]Instituto de Física Teórica - IFT/UNESP, Rua Pamplona 145,01405-900, São Paulo - SP, Brazil

A class of non abelian affine Toda models is constructed in terms of the axial and vector gauged WZW model. It is shown that the multivacua structure of the potential together with non abelian nature of the zero grade subalgebra allows soliton solutions with non trivial electric and topological charges. Their zero curvature representation and the classical r-matrix are also constructed in order to prove their classical integrability.

1 Introduction

The abelian affine Toda field theories provide a large class of integrable models in two dimensions associated to an affine Lie algebra $\tilde{\mathcal{G}}$ (loop algebra) admiting solitons solutions. The Toda fields are defined to parametrize a finite dimensional abelian manifold (Cartan subalgebra of $\tilde{\mathcal{G}}$) and their solitonic character is a consequence of the infinite dimensional Lie algebraic structure responsible for the multivacua configuration leading to a nontrivial topological structure.

A more general class of affine Toda models is obtained by introducing a non abelian structure to the abelian manifold (Cartan subalgebra of $\tilde{\mathcal{G}}$) parametrized by the Toda fields. A systematic manner in classifying the Toda models [1] is in terms of a grading operator Q that decomposes the affine lie algebra $\tilde{\mathcal{G}} = \oplus \mathcal{G}_i$, where the graded subspaces are defined by $[Q, \mathcal{G}_i] = i\mathcal{G}_i$. The Toda fields are defined to parametrize the zero grade subspace $\mathcal{G}_0 \subset \mathcal{G}$.

In this note we discuss a systematic construction of the simplest affine non abelian Toda models associated to $\mathcal{G}_0 = SL(2) \otimes U(1)^{r-1}$ defined by the gradation $Q = h'\hat{d} + \sum_{i \neq a} \frac{2\lambda_i \cdot H}{\alpha_i^2}$, where $\hat{d}, \lambda_i$, α_i are the derivation operator, fundamental weights and simple roots respectively and h' is a real number specified latter. More general and interesting structures arises when the Toda fields lie in the coset $\mathcal{G}_0/\mathcal{G}_0^0$ [2] where $\mathcal{G}_0^0 \in \mathcal{G}_0$ is some specific element of $\mathcal{G}_0$. The simplest model associated to $\mathcal{G}_0/\mathcal{G}_0^0 = SL(2)/U(1)$, is the Lund-Regge (or complex sine-Gordon) model [3].

In section 2 we construct the general action for the affine non abelian Toda models associated to the coset $\frac{SL(2)\otimes U(1)^{r-1}}{U(1)}$ in terms of the gauged two-loop Wess-Zumino-Witten (WZW) model and explicitly show that the non abelian structure of $\mathcal{G}_0$ allows a global gauge invariance responsible for the conservation of an electric charge. In general, the models admit electric (Noether) and magnetic (topological) charges and for imaginary coupling constant β ($\beta^2 = -\frac{2\pi}{k}$) they admit electrically charged topological solitons constructed in [8]. It was shown in [5] that there are two inequivalent ways to gauge fix the $\mathcal{G}_0^0$ degree of freedom, *axial* or *vector*, leading to a pair of actions T-dual to each other.

In section 3 we construct systematically the zero curvature representation for the affine non abelian Toda models, showing therefore, the existence of infinite number of conserved

[1]talk given at the XXIII International Colloquium on Group Theoretical Methods in Physics, Dubna, August 2000

charges. We next construct the classical r-matrix and derive the fundamental Poisson bracket relation which ensures the involution of the conserved charges.

2 Construction of the Model

The generic NA Toda models are classified according to a $\mathcal{G}_0 \subset \mathcal{G}$ embedding induced by the grading operator Q decomposing an finite or infinite Lie algebra $\mathcal{G} = \oplus_i \mathcal{G}_i$ where $[Q, \mathcal{G}_i] = i\mathcal{G}_i$ and $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$. A group element g can then be written in terms of the Gauss decomposition as

$$g = NBM \tag{2.1}$$

where $N = \exp \mathcal{G}_<$, $B = \exp \mathcal{G}_0$ and $M = \exp \mathcal{G}_>$. The physical fields lie in the zero grade subgroup B and the models we seek correspond to the coset $H_-\backslash G/H_+$, for $H_\pm$ generated by positive/negative grade operators.

For consistency with the hamiltonian reduction formalism, the phase space of the G-invariant WZNW model is reduced by specifying the constant generators $\epsilon_\pm$ of grade ± 1. In order to derive an action for B, invariant under

$$g \longrightarrow g' = \alpha_- g \alpha_+, \tag{2.2}$$

where $\alpha_\pm(z, \bar{z})$ lie in the positive/negative grade subgroup we have to introduce a set of auxiliary gauge fields $A \in \mathcal{G}_<$ and $\bar{A} \in \mathcal{G}_>$ transforming as

$$A \longrightarrow A' = \alpha_- A \alpha_-^{-1} + \alpha_- \partial \alpha_-^{-1}, \qquad \bar{A} \longrightarrow \bar{A}' = \alpha_+^{-1} \bar{A} \alpha_+ + \bar{\partial} \alpha_+^{-1} \alpha_+. \tag{2.3}$$

The resulting action is the $G/H (= H_-\backslash G/H_+)$ gauged WZNW

$$\begin{aligned} S_{G/H}(g, A, \bar{A}) &= S_{WZNW}(g) \\ &- \frac{k}{2\pi} \int d^2x Tr \left(A(\bar{\partial} g g^{-1} - \epsilon_+) + \bar{A}(g^{-1}\partial g - \epsilon_-) + A g \bar{A} g^{-1} \right). \end{aligned}$$

Since the action $S_{G/H}$ is H-invariant, we may choose $\alpha_- = N^{-1}$ and $\alpha_+ = M^{-1}$. From the orthogonality of the graded subpaces, i.e. $Tr(\mathcal{G}_i \mathcal{G}_j) = 0, i + j \neq 0$, we find

$$\begin{aligned} S_{G/H}(g, A, \bar{A}) &= S_{G/H}(B, A', \bar{A}') \\ &= S_{WZNW}(B) - \frac{k}{2\pi} \int d^2x Tr[-A'\epsilon_+ - \bar{A}'\epsilon_- + A' B \bar{A}' B^{-1}], \end{aligned} \tag{2.4}$$

where

$$S_{WZNW} = -\frac{k}{4\pi} \int d^2x Tr(g^{-1}\partial g g^{-1} \bar{\partial} g) + \frac{k}{24\pi} \int_D \epsilon^{ijk} Tr(g^{-1}\partial_i g g^{-1} \partial_j g g^{-1} \partial_k g) d^3x, \tag{2.5}$$

and the topological term denotes a surface integral over a ball D identified as space-time.

Performing the integration over the auxiliary fields A and $\bar{A}$, the functional integral

$$Z_\pm = \int DAD\bar{A} \exp(F_\pm), \tag{2.6}$$

where

$$F_\pm = -\frac{k}{2\pi} \int \left(Tr(A - B\epsilon_- B^{-1}) B (\bar{A} - B^{-1}\epsilon_+ B) B^{-1} \right) d^2x \tag{2.7}$$

yields the effective action

$$S = S_{WZNW}(B) + \frac{k}{2\pi}\int Tr\left(\epsilon_+ B\epsilon_- B^{-1}\right) d^2x \tag{2.8}$$

The action (2.8) describe integrable perturbations of the $\mathcal{G}_0$-WZNW model. Those perturbations are classified in terms of the possible constant grade ± 1 operators $\epsilon_\pm$.

More interesting cases arises in connection with non abelian embeddings $\mathcal{G}_0 \subset \mathcal{G}$. In particular, if we supress one of the fundamental weights from Q, the zero grade subspace $\mathcal{G}_0$, acquires a nonabelian structure $sl(2) \otimes u(1)^{rank\mathcal{G}-1}$. Let us consider for instance $Q = h'd + \sum_{i\neq a}^{r} \frac{2\lambda_i \cdot H}{\alpha_i^2}$, where $h' = 0$ or $h' \neq 0$ corresponding to the Conformal or Affine nonabelian (NA) Toda respectively. The absence of λ_a in Q prevents the contribution of the simple root step operator $E_{\alpha_a}^{(0)}$ in constructing ϵ_+. It in fact, allows for reducing the phase space even further. This fact can be understood by enforcing the nonlocal constraint $J_{Y\cdot H} = \bar{J}_{Y\cdot H} = 0$ where Y is such that $[Y \cdot H, \epsilon_\pm] = 0$ and $J = g^{-1}\partial g$ and $\bar{J} = -\bar{\partial} g g^{-1}$. Those generators of $\mathcal{G}_0$ commuting with $\epsilon_\pm$ define a subalgebra $\mathcal{G}_0^0 \subset \mathcal{G}_0$. Such subsidiary constraint is incorporated into the action by requiring symmetry under [9]

$$g \longrightarrow g' = \alpha_0 g \alpha_0' \tag{2.9}$$

where we shall consider $\alpha_0' = \alpha_0(z,\bar{z}) \in \mathcal{G}_0^0$, i.e., *axial symmetry* (the *vector* gauging is obtained by choosing $\alpha_0' = {\alpha_0}^{-1}(z,\bar{z}) \in \mathcal{G}_0^0$). Auxiliary gauge fields $A_0 = a_0 Y \cdot H$ and $\bar{A}_0 = \bar{a}_0 Y \cdot H \in \mathcal{G}_0^0$ are introduced to construct an invariant action under transformations (2.9)

$$\begin{aligned} S(B, A_0, \bar{A}_0) &= S(g_0^f, A'_0, \bar{A}'_0) = S_{WZNW}(B) + \frac{k}{2\pi}\int Tr\left(\epsilon_+ B\epsilon_- B^{-1}\right) d^2x \\ &- \frac{k}{2\pi}\int Tr\left(A_0\bar{\partial}BB^{-1} + \bar{A}_0 B^{-1}\partial B + A_0 B\bar{A}_0 B^{-1} + A_0\bar{A}_0\right) d^2x \end{aligned} \tag{2.10}$$

where the auxiliary fields transform as

$$A_0 \longrightarrow A_0' = A_0 - \alpha_0^{-1}\partial\alpha_0, \qquad \bar{A}_0 \longrightarrow \bar{A}_0' = \bar{A}_0 - \bar{\partial}\alpha_0'(\alpha_0')^{-1}.$$

Such residual gauge symmetry allows us to eliminate an extra field associated to $Y \cdot H$. Notice that the physical fields g_0^f lie in the coset $\mathcal{G}_0/\mathcal{G}_0^0 = (sl(2) \otimes u(1)^{rank\mathcal{G}-1})/u(1)$ of dimension $rank\mathcal{G} + 1$ and are classified according to the gradation Q. It therefore follows that $S(B, A_0, \bar{A}_0) = S(g_0^f, A_0', \bar{A_0}')$.

In [2] a detailed study of the gauged WZNW construction for finite dimensional Lie algebras leading to Conformal NA Toda models was presented. The study of its symmetries was given in ref. [9]. Here we generalize the construction of ref. [2] to infinite dimensional Lie algebras leading to NA Affine Toda models characterized by the broken conformal symmetry and by the presence of solitons.

Consider the Kac-Moody algebra $\widehat{\mathcal{G}}$

$$[T_m^a, T_n^b] = f^{abc}T_{m+n}^c + \hat{c}m\delta_{m+n}\delta^{ab}$$

$$[\hat{d}, T_n^a] = nT_n^a; \quad [\hat{c}, T_n^a] = [\hat{c}, \hat{d}] = 0 \tag{2.11}$$

The NA Toda models we shall be constructing are associated to gradations of the type $Q_a(h') = h'_a d + \sum_{i\neq a}^r \frac{2\lambda_i \cdot H^{(0)}}{\alpha_i^2}$, where h'_a is chosen such that the gradation, $Q_a(h')$, acting on infinite dimensional Lie algebra $\hat{\mathcal{G}}$ ensures that the zero grade subgroup $\mathcal{G}_0$ coincides with its counterpart obtained with $Q_a(h'=0)$ acting on the Lie algebra $\mathcal{G}$ of finite dimension apart from two comuting generators $\hat{c}$ and $\hat{d}$. Since they commute with $\mathcal{G}_0$, the kinetic part decouples such that the conformal and the affine singular NA-Toda models differ only by the potential term characterized by $\hat{\epsilon}_\pm$.

The integration over the auxiliary gauge fields A and $\bar{A}$ require explicit parametrization of B.

$$B = \exp(\tilde{\chi} E_{-\alpha_a}^{(0)}) \exp(R \sum_{i=1}^r Y_i H_i^{(0)} + \Phi(H) + \nu\hat{c} + \eta\hat{d}) \exp(\tilde{\psi} E_{\alpha_a}^{(0)}) \tag{2.12}$$

where $\Phi(H) = \sum_{j=1}^r \sum_{i=2}^r \varphi_i X_i^j H_j^{(0)}$, where $\sum_{j=1}^r Y_j X_i^j = 0, i = 2, \cdots, r$. After gauging away the nonlocal field R, the factor group element becomes

$$g_0^f = \exp(\chi E_{-\alpha_a}^{(0)}) \exp(\Phi(H) + \nu\hat{c} + \eta\hat{d}) \exp(\psi E_{\alpha_a}^{(0)}) \tag{2.13}$$

where $\chi = \tilde{\chi} e^{\frac{1}{2}(Y\cdot\alpha_a)R}, \quad \psi = \tilde{\psi} e^{\frac{1}{2}(Y\cdot\alpha_a)R}$ We therefore get for the zero grade component

$$\begin{aligned} F_0 &= -\frac{k}{2\pi}\int Tr\left(A_0\bar{\partial} g_0^f (g_0^f)^{-1} + \bar{A}_0 (g_0^f)^{-1}\partial g_0^f + A_0 g_0^f \bar{A}_0 (g_0^f)^{-1} + A_0\bar{A}_0\right) d^2x \\ &= -\frac{k}{2\pi}\int \left(a_0\bar{a}_0 2Y^2\Delta - 2(\frac{\alpha_a\cdot Y}{\alpha_a^2})(\bar{a}_0\psi\partial\chi + a_0\chi\bar{\partial}\psi)e^{\Phi(\alpha_a)}\right) d^2x \end{aligned} \tag{2.14}$$

where $\Delta = 1 + \frac{(Y\cdot\alpha_a)^2}{2Y^2}\psi\chi e^{\Phi(\alpha_a)}$, $[\Phi(H), E_{\alpha_a}^{(0)}] = \Phi(\alpha_a) E_{\alpha_a}^{(0)}$.

The effective action is obtained by integrating over the auxiliary fields $A_0, \bar{A}_0$,

$$Z_0 = \int DA_0 D\bar{A}_0 \exp(F_0) \tag{2.15}$$

The total action (2.10) is therefore given as

$$S = -\frac{k}{4\pi}\int \left(Tr(\partial\Phi(H)\bar{\partial}\Phi(H)) + \frac{2\bar{\partial}\psi\partial\chi}{\Delta} e^{\Phi(\alpha_a)} + \partial\eta\bar{\partial}\nu + \partial\nu\bar{\partial}\eta - 2Tr(\hat{\epsilon_+} g_0^f \hat{\epsilon_-} (g_0^f)^{-1})\right) d^2x \tag{2.16}$$

Note that the second term in (2.16) contains both symmetric and antisymmetric parts:

$$\frac{e^{\Phi(\alpha_a)}}{\Delta}\bar{\partial}\psi\partial\chi = \frac{1}{4}\frac{e^{\Phi(\alpha_a)}}{\Delta}(g^{\mu\nu}\partial_\mu\psi\partial_\nu\chi + \epsilon^{\mu\nu}\partial_\mu\psi\partial_\nu\chi), \tag{2.17}$$

where $g_{\mu\nu}$ is the 2-D metric of signature $g_{\mu\nu} = diag(1, -1)$, $z = t + x \quad \bar{z} = t - x$. For $n = 1$ ($\mathcal{G} \equiv A_1$, $\Phi(\alpha_1)$ is zero) the antisymmetric term is a total derivative:

$$\epsilon^{\mu\nu}\frac{\partial_\mu\psi\partial_\nu\chi}{1+\psi\chi} = \frac{1}{2}\epsilon^{\mu\nu}\partial_\mu\left(\ln\{1+\psi\chi\}\partial_\nu \ln\frac{\chi}{\psi}\right), \tag{2.18}$$

and it can be neglected. This A_1-NA-Toda model (in the conformal case), is known to describe the 2-D black hole solution for (2-D) string theory [7]. The $\mathcal{G}$-NA conformal Toda model can be used in the description of specific (n+1)-dimensional black string theories [10], with n-1-flat and 2-non flat directions ($g^{\mu\nu}G_{ab}(X)\partial_\mu X^a \partial_\nu X^b$, $X^a = (\psi, \chi, \varphi_i)$), containing axions ($\epsilon^{\mu\nu}B_{ab}(X)\partial_\mu X^a \partial_\nu X^b$) and tachyons ($\exp\{-k_{ij}\varphi_j\}$), as well. The affine A_1-NA Toda theory with $\epsilon_\pm = H^{(\pm)}$ correspond to the Lund-Regge model describing charged solitons.

It is clear that the presence of the $e^{\Phi(\alpha_a)}$ in (2.16) is responsible for the antisymmetric tensor generating CPT breaking terms. On the other hand, notice that $\Phi(\alpha_a)$ depend upon the subsidiary nonlocal constraint $J_{Y\cdot H} = \bar{J}_{Y\cdot H} = 0$ and hence upon the choice of the vector Y. It is defined to be orthogonal to all roots contained in $\epsilon_\pm$. A Lie algebraic condition for the absence of axionic terms was found in [2] and has provided a construction of a family of torsionless NA Toda models in [5]. All vector models are CPT invariant by construction.

The action (2.16) is invariant under the global $U(1)$ transformation

$$\psi \to e^{i\epsilon}\psi, \qquad \chi \to e^{-i\epsilon}\chi \tag{2.19}$$

The corresponding Noether current is

$$J^\mu = -\frac{ik}{8\pi}\frac{e^{\Phi(\alpha_a)}}{\Delta}\{\psi\left(g^{\nu\mu}\partial_\nu\chi - \epsilon^{\nu\mu}\partial_\nu\chi\right) - \chi\left(g^{\nu\mu}\partial_\nu\psi + \epsilon^{\nu\mu}\partial_\nu\psi\right)\} \tag{2.20}$$

and the electric charge is given in terms of the nonlocal field R defined below in (3.28) as

$$Q = \int J_o dx = -\frac{ik}{4\pi}\left(\frac{r}{r+1}\right)\left[R(x\to\infty) - R(x\to-\infty)\right]. \tag{2.21}$$

Apart from the global $U(1)$ symmetry (2.19) there is a discrete set of field transformations leaving the action (2.16) unchanged. Such transformations (for imaginary β, $\beta \to i\beta_0$) give rise to multivacua configuration and hence to nontrivial topological charges

$$\begin{aligned} Q_j &= \int_{-\infty}^{+\infty} J_j^0 dx, \quad J_j^\mu = -i\frac{2r}{\beta}\epsilon^{\mu\nu}\partial_\nu\varphi_j, \quad j = 2, \cdots r \\ Q_\theta &= \int_{-\infty}^{+\infty} J_\theta^0 dx \quad J_\theta^\mu = -i\frac{1}{2\beta^2}\epsilon^{\mu\nu}\partial_\nu ln\left(\frac{\chi}{\psi}\right) \end{aligned} \tag{2.22}$$

Let us explicitly consider the $A_r^{(1)}$ model described by the Lagrangean density (4.36) (with fields rescaled by $\varphi_i \to \beta\varphi_i, \chi \to \beta\chi, \psi \to \beta\psi$, $\beta^2 = -\frac{2\pi}{k}$) invariant under the following set of discrete transformations,

$$\varphi_j' = \varphi_j + \frac{2\pi(j-1)N}{\beta_0 r}, j = 2, \cdots r, \quad \chi' = e^{i\pi(\frac{N}{r}+s_2)}\chi, \quad \psi' = e^{i\pi(\frac{N}{r}+s_1)}\psi \tag{2.23}$$

where s_1, s_2 are both even or odd integers and the following CP transformations (P: $x \to -x$)

$$\varphi_j'' = \varphi_j, j = 2, \cdots r, \quad \chi'' = \psi, \quad \psi'' = \chi \tag{2.24}$$

The minimum of the potential (for the choice $\eta = 0$) corresponds to the following field configuration

$$\varphi_j^{(N)} = \frac{2\pi(j-1)N}{\beta_0 r}, \quad \theta^{(L)} = \frac{1}{2i\beta_0} ln\left(\frac{\chi}{\psi}\right) = \frac{\pi L}{2\beta_0}, \quad \rho^{(0)} = 0 \quad j = 2, \cdots r \tag{2.25}$$

where N, L are arbitrary integers, and the new fields θ and ρ are defined as

$$\psi = \frac{1}{\beta_0} e^{i\beta_0(\frac{1}{2}\varphi_2 - \theta)} \sinh(\beta_0 \rho), \quad \chi = \frac{1}{\beta_0} e^{i\beta_0(\frac{1}{2}\varphi_2 + \theta)} \sinh(\beta_0 \rho) \tag{2.26}$$

In fact eqns, (2.25) also represent constant solutions of the eqns. of motion (3.32)-(3.35) which allows us to derive the values of the topological charge (2.22): $Q_j = \frac{4\pi}{\beta_0^2}(j-1)(N_+ - N_-)$, $Q_\theta = \frac{2\pi}{\beta_0^2}(L_+ - L_-)$

3 Zero Curvature and Equations of Motion

The equations of motion for the NA Toda models are known to be of the form [1]

$$\bar{\partial}(B^{-1}\partial B) + [\hat{\epsilon_-}, B^{-1}\hat{\epsilon_+} B] = 0, \quad \partial(\bar{\partial} B B^{-1}) - [\hat{\epsilon_+}, B\hat{\epsilon_-} B^{-1}] = 0 \tag{3.27}$$

The subsidiary constraint $J_{Y \cdot H^{(0)}} = Tr(B^{-1}\partial B Y \cdot H^{(0)})$ and $\bar{J}_{Y \cdot H^{(0)}} = Tr(\bar{\partial} B B^{-1} Y \cdot H^{(0)}) = 0$ can be consistently imposed since $[Y \cdot H^{(0)}, \hat{\epsilon_\pm}] = 0$ as can be obtained from (3.27) by taking the trace with $Y.H^{(0)}$. Solving those equations for the nonlocal field R yields,

$$\partial R = (\frac{Y \cdot \alpha_a}{Y^2}) \frac{\psi \partial \chi}{\Delta} e^{\Phi(\alpha_a)}, \qquad \bar{\partial} R = (\frac{Y \cdot \alpha_a}{Y^2}) \frac{\chi \bar{\partial} \psi}{\Delta} e^{\Phi(\alpha_a)} \tag{3.28}$$

The equations of motion for the fields ψ, χ and $\varphi_i, i = 2, \cdots, r$ obtained from (3.27) after imposing the subsidiary constraints (3.28) coincide precisely with the Euler-Lagrange equations derived from (2.16). Alternatively, (3.27) admits a zero curvature representation $\partial \bar{A} - \bar{\partial} A + [A, \bar{A}] = 0$ where

$$A = B\hat{\epsilon_-} B^{-1}, \quad \bar{A} = -\hat{\epsilon_+} - \bar{\partial} B B^{-1} \tag{3.29}$$

Whenever the constraints (3.28) are incorporated into A and $\bar{A}$ in (3.29), equations (3.27) yields the zero curvature representation of the NA singular Toda models.

We shall be considering $\hat{\mathcal{G}} = A_r^{(1)}$, $Q = r\hat{d} + \sum_{i=2}^r 2\frac{\lambda_i \cdot H^{(0)}}{\alpha_i^2}$, $\sum_{i=1}^r Y_i H_i^{(0)} = 2\frac{\lambda_1 \cdot H^{(0)}}{\alpha_1^2}$, $\sum_{j=1}^r X_i^j H_j^{(0)} = h_i^{(0)} = \frac{2\alpha_i \cdot H^{(0)}}{\alpha_i^2}$ and $\hat{\epsilon_\pm} = \mu \left(\sum_{i=2}^r E_{\pm\alpha_i}^{(0)} + E_{\mp(\alpha_2 + \cdots + \alpha_r)}^{(\pm 1)} \right)$, $\mu > 0$.

Using the explicit parametrization of B given in (2.12), we find, in a systematic manner, the following form for A and $\bar{A}$

$$\begin{aligned} A &= \mu(\sum_{i=2}^r e^{-\sum_{j=2}^r K_{i,j}\varphi_j} E_{-\alpha_i}^{(0)} - \chi e^{-\frac{1}{2}R - 2\varphi_2 + \varphi_3} E_{-\alpha_1 - \alpha_2}^{(0)} \\ &+ \psi e^{\frac{1}{2}R + \varphi_r - \eta} E_{\alpha_1 + \cdots \alpha_r}^{(-1)} + (1 + \psi\chi e^{-\varphi_2}) E_{\alpha_2 + \cdots + \alpha_r}^{(-1)} e^{\varphi_2 + \varphi_r - \eta}) \end{aligned} \tag{3.30}$$

and

$$\overline{A} = \mu\left(-\sum_{i=2}^{r} E_{\alpha_i}^{(0)} - E_{-\alpha_2-\ldots-\alpha_r}^{(1)}\right) - \left(\overline{\partial}\chi - \chi\overline{\partial}\varphi_2 + (\frac{1}{2\lambda_1^2} - 1)\frac{\chi^2\overline{\partial}\psi}{\Delta}e^{-\varphi_2}\right)e^{-\frac{1}{2}R}E_{-\alpha_1}^{(0)}$$

$$-\frac{\overline{\partial}\psi}{\Delta}e^{\frac{1}{2}R-\varphi_2}E_{\alpha_1}^{(0)} - \overline{\partial}\nu\hat{c} - \overline{\partial}\eta\hat{d} - \sum_{i=2}^{r}\overline{\partial}\varphi_i h_i^{(0)} - \frac{\chi\overline{\partial}\psi}{\Delta}e^{-\varphi_2}\sum_{j=2}^{r}\left(\frac{r+1-j}{r}\right)h_j^{(0)}, \qquad (3.31)$$

leading to the following equations of motion

$$\partial\overline{\partial}\eta = 0, \quad \partial\overline{\partial}\nu = \mu^2 e^{\varphi_r - \eta}(e^{\varphi_2} + \psi\chi), \qquad (3.32)$$

$$\partial\left(\frac{e^{-\varphi_2}\overline{\partial}\psi}{\Delta}\right) + \left(\frac{r+1}{2r}\right)\frac{\psi e^{-2\varphi_2}\partial\chi\overline{\partial}\psi}{\Delta^2} + \mu^2 e^{\varphi_r-\eta}\psi = 0, \qquad (3.33)$$

$$\overline{\partial}\left(\frac{e^{-\varphi_2}\partial\chi}{\Delta}\right) + \left(\frac{r+1}{2r}\right)\frac{\chi e^{-2\varphi_2}\partial\chi\overline{\partial}\psi}{\Delta^2} + \mu^2 e^{\varphi_r-\eta}\chi = 0 \qquad (3.34)$$

$$\begin{aligned}\partial\bar{\partial}\varphi_i \quad &+ \quad (\frac{r+1-i}{r})\frac{\partial\chi\bar{\partial}\psi e^{-\varphi_2}}{\Delta^2} \\ &+ \quad \mu^2 e^{\varphi_2+\varphi_r-\eta}\left(1 + (\frac{i-1}{r})\psi\chi e^{-\varphi_2}\right) - \mu^2 e^{-\sum_{j=2}^{r} K_{ij}\varphi_j} = 0, \qquad (3.35)\end{aligned}$$

$i = 2, \cdots r$, where we have normalized $\alpha^2 = 2$.

4 Classical r-matrix

Consider the effective action for the conformal affine A_r-NA Toda model specified in the previous section by the action

$$\begin{aligned}S_{eff} \quad &= \quad -\frac{k}{4\pi}\int d^2x\left(\sum_{i,j=2}^{r} K_{i,j}\partial\varphi_i\overline{\partial}\varphi_j + \partial\nu\overline{\partial}\eta + \partial\eta\overline{\partial}\nu\right. \\ &+ \quad \left.\frac{2e^{-\varphi_2}\partial\chi\overline{\partial}\psi}{\Delta} - 2\mu^2\left(\sum_{i=2}^{r} e^{-\sum_{j=2}^{r} K_{i,j}\varphi_j} + e^{\varphi_r+\varphi_2-\eta}(1+\psi\chi e^{-\varphi_2})\right)\right). \qquad (4.36)\end{aligned}$$

where $g^{00} = -g^{11} = 1$ and $K_{i,j} = Tr(h_i^{(0)}h_j^{(0)})$. The canonical momenta are

$$\Pi_{\varphi_k} = -\frac{k}{8\pi}K_{k,i}\partial_t\varphi_i, \;\; k = 2, \cdots, r \quad \Pi_\nu = -\frac{k}{8\pi}\partial_t\eta, \quad \Pi_\eta = -\frac{k}{8\pi}\partial_t\nu, \qquad (4.37)$$

$$\Pi_\chi = -\frac{k}{4\pi}\frac{e^{-\varphi_2}\overline{\partial}\psi}{\Delta} \qquad \Pi_\psi = -\frac{k}{4\pi}\frac{e^{-\varphi_2}\partial\chi}{\Delta}. \qquad (4.38)$$

Let $A = \frac{1}{2}(A_0 + A_1)$ and $\bar{A} = \frac{1}{2}(A_0 - A_1)$ given by eqns. (3.30) and (3.31). Consider the gauge transformation

$$A'_\mu = SA_\mu S^{-1} - \partial_\mu SS^{-1} \qquad (4.39)$$

where $S = S_3S_2S_1$ and

$$S_1 = \exp\left(-\frac{\varphi_1}{2}2\frac{\lambda_1\cdot H^{(0)}}{\alpha_1^2} - \frac{\nu}{2}\hat{c} - \frac{\eta}{2}\hat{d} - \sum_{i=3}^{r}\frac{\varphi_i}{2}h_i^{(0)}\right), \tag{4.40}$$

$$S_2 = \exp(-\chi E_{-\alpha_1}^{(0)}), \quad S_3 = \exp(-\frac{\varphi_2}{2}h_2^{(0)}) \tag{4.41}$$

yielding

$$\begin{aligned} A'_x &= -\frac{4\pi}{k}\left(\Pi_\eta \hat{c} + \Pi_\nu \hat{d}\right) + \mu\sum_{i=2}^{r} e^{-\frac{1}{2}\sum_{j=2}^{r}K_{i,j}\varphi_j}(E_{\alpha_i}^{(0)} + E_{-\alpha_i}^{(0)}) \\ &+ \mu e^{\frac{1}{2}(\varphi_r-\eta)}(\psi E_{\alpha_1+\ldots+\alpha_r}^{(-1)} + \chi E_{-(\alpha_1+\ldots+\alpha_r)}^{(1)}) + \mu e^{\frac{1}{2}(\varphi_2+\varphi_r-\eta)}(E_{\alpha_2+\ldots+\alpha_r}^{(-1)} + E_{-(\alpha_2+\ldots+\alpha_r)}^{(1)}) \\ &- \frac{4\pi}{k}e^{\frac{1}{2}\varphi_2}(\Pi_\chi E_{\alpha_1}^{(0)} + \Pi_\psi E_{-\alpha_1}^{(0)}) - \frac{4\pi}{k}\sum_{l,k=1}^{r-1}(\tilde{K}^{-1})_{l,k}\Pi_{\varphi_{k+1}}h_{l+1}^{(0)} \\ &- \frac{2\pi}{k}(\frac{r+1}{r})(\psi\Pi_\psi + \chi\Pi_\chi)\frac{2\lambda_1\cdot H^{(0)}}{\alpha_1^2} \end{aligned} \tag{4.42}$$

where $\tilde{K}$ denotes the matrix defined by K removing the first row and first column. The fundamental Poisson bracket relation (FPR)

$$\{A'_x(y,t)\otimes A'_x(z,t)\}_{PB} = [r, A'_x(y,t)\otimes I + I\otimes A'_x(z,t)]\delta(y-z), \tag{4.43}$$

can then be verified, where the l.h.s. is evaluated using the canonical comutation relations and r denotes the classical r-matrix

$$r = -\frac{2\pi}{k}[C^+ - C^-], \tag{4.44}$$

where

$$C^+ = \sum_{m=1}^{\infty}\sum_{a,b=1}^{r}\frac{\alpha_b^2}{2}(K^{-1})_{a,b}\left(h_a^{(m)}\otimes h_b^{(-m)}\right) + \frac{1}{2}\sum_{\alpha>0}\frac{\alpha^2}{2}\left(E_\alpha^{(0)}\otimes E_{-\alpha}^{(0)}\right) + \tag{4.45}$$

$$+\sum_{m=1}^{\infty}\sum_{\alpha>0}\frac{\alpha^2}{2}\left[E_\alpha^{(m)}\otimes E_{-\alpha}^{(-m)} + E_{-\alpha}^{(m)}\otimes E_\alpha^{(-m)}\right], \tag{4.46}$$

$$C^- = \sigma C^+, \quad \text{where} \quad \sigma(A_1\otimes B_1)\ldots(A_n\otimes B_n) = (B_1\otimes A_1)\ldots(B_n\otimes A_n). \tag{4.47}$$

and

$$C_0 = \sum_{a,b=1}^{r}K_{ab}^{-1}h_a^{(0)}\otimes h_b^{(0)} + \hat{c}\otimes\hat{d} + \hat{d}\otimes\hat{c}. \tag{4.48}$$

In order to evaluate the r.h.s. of eqn. (4.43) we follow the arguments of ref. [4]. Let

$$\mathcal{C} = \sum_{m=-\infty}^{\infty}\sum_{a,b=1}^{r}K_{ab}^{-1}T_a^m\otimes T_b^{-m} + \hat{c}\otimes\hat{d} + \hat{d}\otimes\hat{c} \tag{4.49}$$

be the Casimir operator satisfying $[\mathcal{C}, 1\otimes T + T\otimes 1] = 0$ It then follows from direct calculation that

$$[C_\pm, 1\otimes h_a^{(0)} + h_a^{(0)}\otimes 1] = [C_\pm, 1\otimes\hat{d} + \hat{d}\otimes 1] = [C_\pm, 1\otimes\hat{c} + \hat{c}\otimes 1] = 0 \tag{4.50}$$

$$
\begin{aligned}
[C_0, 1 \otimes E_\beta^{(n)} + E_\beta^{(n)} \otimes 1] &= \left(h_\beta^{(0)} \otimes E_\beta^{(n)} + E_\beta^{(n)} \otimes h_\beta^{(0)}\right) + n\left(\hat{c} \otimes E_\beta^{(n)} + E_\beta^{(n)} \otimes \hat{c}\right) \\
&= -[C_+ + C_-, 1 \otimes E_\beta^{(n)} + E_\beta^{(n)} \otimes 1] \qquad (4.51)
\end{aligned}
$$

where β denotes an arbitrary simple root. Using the fact that the sum of a positive (negative) simple root with a negative (positive) root is never a positive (negative) root we obtain,

$$
\begin{aligned}
[C_+, 1 \otimes E_\beta^{(n)} + E_\beta^{(n)} \otimes 1] &= -\left(E_\beta^{(n)} \otimes h_\beta^{(0)} + X_+ \otimes X_-\right) \\
[C_-, 1 \otimes E_\beta^{(n)} + E_\beta^{(n)} \otimes 1] &= -\left(h_\beta^{(0)} \otimes E_\beta^{(n)} + X_- \otimes X_+\right) \qquad (4.52)
\end{aligned}
$$

where $X_\pm$ denote terms with positive (negative) root step operators. By adding these two equations and comparing with (4.51), we conclude that the last terms in the r.h.s. of (4.52) vanish, and therefore

$$
[C_+ - C_-, 1 \otimes E_\beta^{(0)} + E_\beta^{(0)} \otimes 1] = \left(h_\beta^{(0)} \otimes E_\beta^{(0)} - E_\beta^{(0)} \otimes h_\beta^{(0)}\right). \qquad (4.53)
$$

Analogously we find

$$
\begin{aligned}
[C_+ - C_-, 1 \otimes E_{\mp\psi}^{(\pm 1)} &+ E_{\mp\psi}^{(\pm 1)} \otimes 1] = -\left(h_\psi^{(0)} \otimes E_{\mp\psi}^{(\pm 1)} - E_{\mp\psi}^{(\pm 1)} \otimes h_\psi^{(0)}\right) \\
&+ \left(\hat{c} \otimes E_{\mp\psi}^{(\pm 1)} - E_{-\psi}^{(1)} \otimes \hat{c}\right), \quad h_\psi = \psi \cdot H \qquad (4.54)
\end{aligned}
$$

and

$$
\begin{aligned}
& [C_+ - C_-, 1 \otimes E_{\pm(\alpha_2 + \cdots \alpha_r)}^{(\mp 1)} + E_{\pm(\alpha_2 + \cdots + \alpha_r)}^{(\mp 1)} \otimes 1] \\
= & -\sum_{a=2}^{r} \left(h_a^{(0)} \otimes E_{\pm(\alpha_2 + \cdots + \alpha_r)}^{(\mp 1)} - E_{\pm(\alpha_2 + \cdots \alpha_r)}^{(\mp 1)} \otimes h_a^{(0)}\right) \\
+ & \left(\hat{c} \otimes E_{\pm(\alpha_2 + \cdots \alpha_r)}^{(\mp 1)} - E_{\pm(\alpha_2 + \cdots \alpha_r)}^{(\mp 1)} \otimes \hat{c}\right) + 2\left(E_{\pm\psi}^{(\mp 1)} \otimes E_{\mp\alpha_1}^{(0)} - E_{\mp\alpha_1}^{(0)} \otimes E_{\pm\psi}^{(\mp 1)}\right) \qquad (4.55)
\end{aligned}
$$

Using (4.53)-(4.55) we find agreement for both sides of the fundamental Poisson bracket relation (4.43)

5 Remarks on Soliton Solutions

A systematic and elegant method to obtain soliton solutions is to consider the dressing of the gauge connection A and $\bar{A}$ in (3.30) and (3.31) at vacuum configuration [6], i.e. $A_{vac} = \epsilon_-$ and $\bar{A}_{vac} = -\epsilon_+ - \mu^2 z \hat{c}$. A crucial ingredient in classifying the soliton solutions is the Heisenberg subalgebra generated by $\epsilon_\pm$ and its eigenstates (vertex operators). We have considered in [8] the solutions for the $A_r^{(1)}$ model described in section 3 with vertices constructed explicitly in ref. [11]. The solitons were classified according to the vertices and give rise to neutral solutions (solutions of the A_{r-1} affine abelian Toda model with $\psi = \chi = 0$) and nontrivial charged solutions.($\psi = \chi \neq 0$). The composition of such solutions give rise to 2-soliton and breather solutions. These are classified according to *neutral-neutral* (2-solitons of the abelian A_{r-1} affine abelian Toda model), *neutral-charged* and *charged-charged* solutions. The time delays were also considered.

Acknowledgements We are grateful to CNPq and Fapesp for financial support. One of us (JFG) thanks Fapesp and the organizers of the XXIII International Colloquium on Group Theoretical Methods in Physics for financial support and kind hospitality.

References

[1] A. N. Leznov, M. V. Saveliev, Group Theoretical Methods for Integration of Non-linear Dynamical Systems, Progress in Physics, Vol. 15 (1992), Birkhauser Verlag, Berlin; A. N. Leznov and M. V. Saveliev, Commun. Math. Phys. **89** (1983) 59

[2] J.F. Gomes, F.E.M. da Silveira, G.M. Sotkov and A.H. Zimerman, “Singular Non-Abelian Toda Theories”, hepth 9810057, in “Nonassociative Algebras and its Applications”, Lec. Notes in Mathematics, Vol 211,p. 125-136 Ed. R. Costa et. al., Marcel Dekker, (1999). J.F. Gomes, E. P. Gueuvoghlanian, F.E.M. da Silveira, G.M. Sotkov and A.H. Zimerman, “Singular Conformal, and Conformal Affine Non-Abelian Toda Theories”, “M.V. Saveliev Memorial Volume”, Dubna (2000), Ed. A.N. Sissakian, p. 38-50, see also hepth/0002173.

[3] F. Lund, Ann. of Phys. **415** (1978) 251;
F. Lund and T. Regge, Phys. Rev. **D14** (1976) 1524

[4] H. Aratyn, L.A. Ferreira, J.F. Gomes and A.H. Zimerman, Phys. Lett **B 254** (1991) 372

[5] J.F. Gomes, E. P. Gueuvoghlanian ,F.E.M. da Silveira, G.M. Sotkov and A.H. Zimerman, “ T-duality of axial and vector dyonic integrable models, hepth/0007116

[6] L.A. Ferreira, J.L. Miramontes and J.S. Guillen, J. Math. Phys. **38**, (1997), 882-901

[7] E. Witten, Phys. Rev. Lett. **38** (1978) 121

[8] J.F. Gomes, E.P. Gueuvoghlanian, G.M. Sotkov and A.H. Zimerman, IFT-Unesp preprint, IFT-P-064-2000, “Electrically Charged Topological Solitons” also in hepth/0007169; J.F. Gomes, E.P. Gueuvoghlanian, G.M. Sotkov and A.H. Zimerman, IFT-Unesp preprint (in preparation) “Dyonic Integrable Models”

[9] J.F. Gomes, G.M. Sotkov and A.H. Zimerman, Phys. Lett. **435B** (1998) 49, also in hepth/9803122, Ann. of Phys.**274**, (1999), 289-362, also in hepth/9803234

[10] J.-L. Gervais and M. V. Saveliev, Phys. Lett. **286B** (1992) 271

[11] H. Aratyn, L.A. Ferreira, J.F. Gomes and A.H. Zimerman, Supersymmetry and Integrable models (Lecture Notes in Physics Vol. 502), Ed. H. Aratyn et. al. (Berlin, Springer), p. 197-210, (1998); J. Phys. **A31**, (1998) 9483-9492